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A class of acyclic digraphs with interval competition graphs

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Abstract

Let D be an acyclic digraph. The competition graph of D has the same set of vertices as D and an edge between vertices u and v if and only if there is a vertex x in D such that (u, x) and (v, x) are arcs of D. In this paper, we show that the competition graphs of doubly partial orders are interval graphs. We also show that an interval graph together with enough isolated vertices is the competition graph of a doubly partial order. Finally, we introduce a new notion called the doubly partial order competition number of an interval graph and present some open questions.

Keywords: Competition graph; Interval graph; Doubly partial order; Doubly partial order competition number

1. Introduction

Given a digraph $D = (V, A)$, the *competition graph* $G = C(D)$ of D has the same vertex set and has an edge xy if for some vertex $u \in V$, the arcs (x, u) and (y, u) are in D. The notion of the competition graph is due to Cohen [2] and has arisen from ecology. A *food web* in an ecosystem is a digraph whose vertices are the species of the system and which has an arc from vertex u to vertex v if and only if u prey on v. Given a food web F, it is said that species u and v compete if and only if they have a common prey. Competition graphs also have applications in coding, radio transmission, and modelling of complex economic

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systems. (See [16,18] for a summary of these applications and [6] for a sample paper on the modelling application.)

We say a graph G is an *interval graph* if it is the intersection graph of some family of intervals on the real line. Cohen [2–4] observed empirically that most competition graphs of acyclic digraphs representing food webs are interval graphs. Roberts [17] asked whether or not Cohen's observation was just an artifact of the construction and concluded that it was not by showing that if G is an arbitrary graph, then G together with as many isolated vertices as the edges of G is the competition graph of an acyclic digraph D. (Add a vertex i_α corresponding to each edge $\alpha = \{a, b\}$ of G, and draw arcs from a and b to i_α.) He then asked for a characterization of acyclic digraph D whose competition graphs $C(D)$ are interval graphs. Since then the problem has remained elusive and it has been the basic open problem in the study of competition graphs. There have been efforts at settling the problem and some progress has been made. Cohen [4] approached the problem from a statistical point of view, trying to build statistical models for the construction of D so that $C(D)$ is (likely to be) an interval graph. Steif [21] showed that there could be no forbidden subgraph characterization of acyclic digraphs whose competition graphs are interval. Lundgren and Maybee [13] gave some results which characterize such a digraph D. But these results essentially boiled down to calculating $C(D)$ and using one of the well-known (and efficient) characterizations of an interval graph. While this solves the problem, it is not what we want: a characterization in terms of properties of D. Since the general problem of characterizing acyclic digraphs whose competition graphs are interval seems difficult, Hefner et al. [8] attacked it by putting a constraint on both the indegrees and outdegrees of D.

The study on acyclic digraphs whose competition graphs are interval led to several new problems and applications. One of them is to characterize competition graphs of an interesting family of digraphs. There have been a number of papers about competition graphs of specific classes of digraphs. For instance, competition graphs of acyclic digraphs have been studied in [1,19], of arbitrary digraphs with or without loops in [1,12], of strongly connected digraphs in [5], of Hamiltonian digraphs in [5,7], of interval digraphs in [11], for various classes of symmetric digraphs in [14–16], of semiorders, of acyclic digraphs satisfying property $C(p)$, and of acyclic digraphs satisfying property $C^*(p)$ in [10]. By means of extending the results obtained in [10], Roberts (pers. comm.) proposed a problem of characterizing competition graphs of doubly partial orders. While we tried to answer his question, we could show that the competition graphs of doubly partial orders are interval graphs. We also show that an interval graph together with enough isolated vertices is the competition graph of a doubly partial order. The authors believe that these results make a significant contribution to the problem of characterizing acyclic digraphs whose competition graphs are interval.

The vast literature of competition graphs is summarized in the survey paper by Lundgren [12] and Kim [9].

2. Main results

After Kim and Roberts [10] characterized competition graphs of semiorders and interval orders, they introduced acyclic digraphs satisfying $C(p)$ and $C^*(p)$ as generalizations of

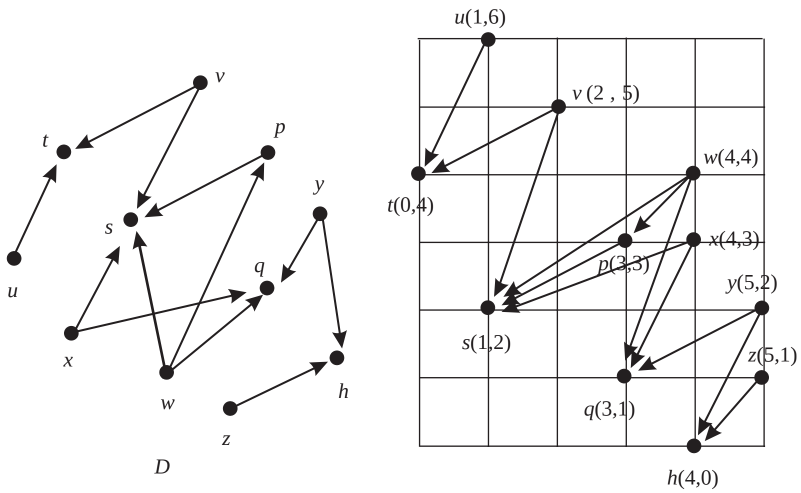

Fig. 1. A doubly partial order D and a doubly partial order on a subset of $\mathbb{R}^2$ which is isomorphic to D.

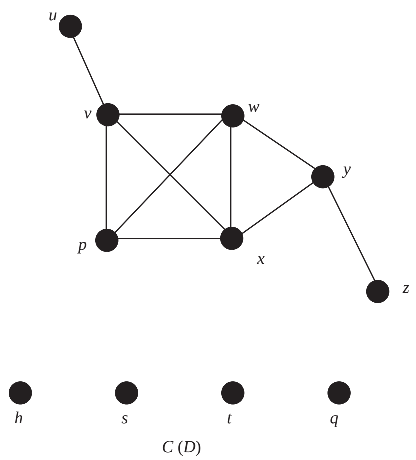

Fig. 2. The competition graph of the doubly partial order D given in Fig. 1.

semiorders. Then Roberts (pers. comm.) proposed a problem of characterizing the competition graphs of doubly partial orders, another generalization of semiorders.

A relation R on a subset S of $\mathbb{R}^2$ is a *doubly partial order on S* if $(x, y)R(z, w)$ whenever $x < z$ and $y < w$ for $(x, y), (z, w) \in S$. A digraph D is called a *doubly partial order* if D is isomorphic to a digraph of a doubly partial order relation R on a subset of $\mathbb{R}^2$ (see Fig. 1 for example).

We say that a graph G is the *competition graph of a doubly partial order* if there is a doubly partial order D such that G is isomorphic to the competition graph of D. (See Fig. 2 for example.)

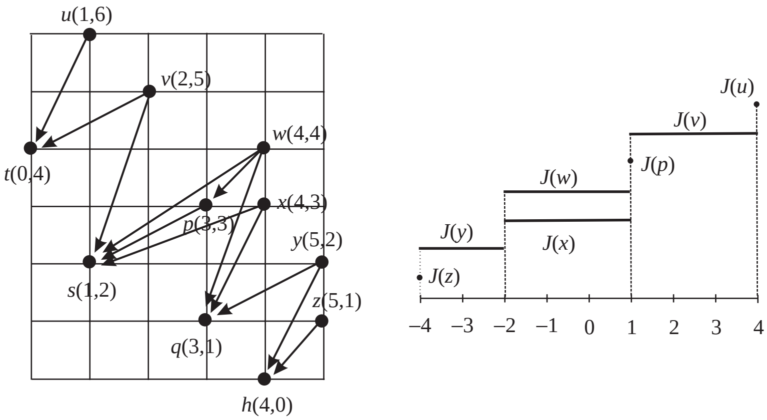

Fig. 3. A doubly partial order and the intervals assigned to vertices with outgoing arcs as instructed in the proof of Theorem 1.

Theorem 1. *Competition graphs of a digraph of a doubly partial order are interval graphs.*

Proof. Let G be the competition graph of a digraph D of doubly partial order on a subset of $\mathbb{R}^2$. We shall give an interval assignment to each vertex so that G is the intersection of the family of those intervals. We may disregard the isolated vertices of G as we can assign an interval to an isolated vertex that does not overlap with any other intervals. In this context, we may ignore vertices with no outgoing arcs. Now given (non-isolated) vertex (x, y) of D, let $l(x, y)$ be the smallest one and $r(x, y)$ be the greatest one among the members of

$$\{b - a \mid (a, b) \text{ is a prey of } (x, y)\}.$$

Now assign closed interval $[l(x, y), r(x, y)]$ to (x, y) (see Fig. 3 for illustration). Since (x, y) is not isolated, there exists a vertex (a, b) such that

$$(a < x) \wedge (b < y).$$

Since $l(x, y) \leqslant b - a \leqslant r(x, y)$, $[l(x, y), r(x, y)] \neq \emptyset$, we will show that the graph obtained by deleting all the isolated vertices from G is the intersection graph of the family of these intervals.

Take two adjacent vertices (x, y) and (z, w) of $C(D)$. Then there exists a vertex (a, b) such that

$$(a < \min\{x, z\}) \wedge (b < \min\{y, w\}).$$

By definition,

$$(\max\{l(x, y), l(z, w)\} \leqslant b - a) \wedge (\min\{r(x, y), r(z, w)\} \geqslant b - a)$$

and therefore

$$b - a \in [l(x, y), r(x, y)] \cap [l(z, w), r(z, w)].$$

Now suppose that two non-isolated vertices (x, y) and (z, w) are non-adjacent. Then $x \leqslant z$ and $y \geqslant w$, or $x \geqslant z$ and $y \leqslant w$. For, otherwise, a prey of one is that of the other and so two vertices are adjacent in $C(D)$, contradiction. We may assume that $x \leqslant z$ and $y \geqslant w$. Suppose that there exists a real number t belonging to $[l(x, y), r(x, y)] \cap [l(z, w), r(z, w)]$. We will reach a contradiction. By definition, there exists prey (a, b), (c, d) of (x, y) such that

$$b - a \leqslant t \leqslant d - c$$

and prey (e, f) and (g, h) of (z, w) such that

$$f - e \leqslant t \leqslant h - g.$$

From the two inequalities above,

$$b - a \leqslant h - g. \tag{1}$$

Since (a, b) and (c, d) are prey of (x, y), $x > \max\{a, c\}$. Similarly, $w > \max\{f, h\}$. Since $y \geqslant w$ and $z \geqslant x$, it is true that $y > \max\{f, h\}$ and $z > \max\{a, c\}$. Suppose that $x \leqslant g$ and $w \leqslant b$. Then $a < g$ and $h < b$ and therefore

$$h - g < b - a,$$

which contradicts inequality (1). Thus, $x > g$ or $w > b$, which implies that either (g, h) or (a, b) is a common prey of (x, y) and (z, w), which is a contradiction. □

Now it is natural to ask if given an interval graph, there is a doubly partial order whose competition graph is G. The answer is yes as long as we are allowed to add enough vertices to G.

Theorem 2. *An interval graph with sufficiently many isolated vertices is the competition graph of a doubly partial order.*

Proof. Take an interval graph G and let J be an interval assignment for the vertices of G. We may assume that all the intervals are open and $J(v) \neq J(w)$ for distinct vertices v and w of G. Without loss of generality, G has no isolated vertices. We define two functions l and r from $V(G)$ to $\mathbb{R}$ as follows: given a vertex v of G, we define $l(v)$ (respectively, $r(v)$) by the coordinate of the left (respectively, right) boundary of $J(v)$. Without loss of generality, we may assume that $\min\{l(x) \mid x \in V(G)\} = 0$. Let

$$M = \max\{r(x) \mid x \in V(G)\}.$$

Now we assign an ordered pair to each vertex v of $V(G)$

$$(r(v), M - l(v)).$$

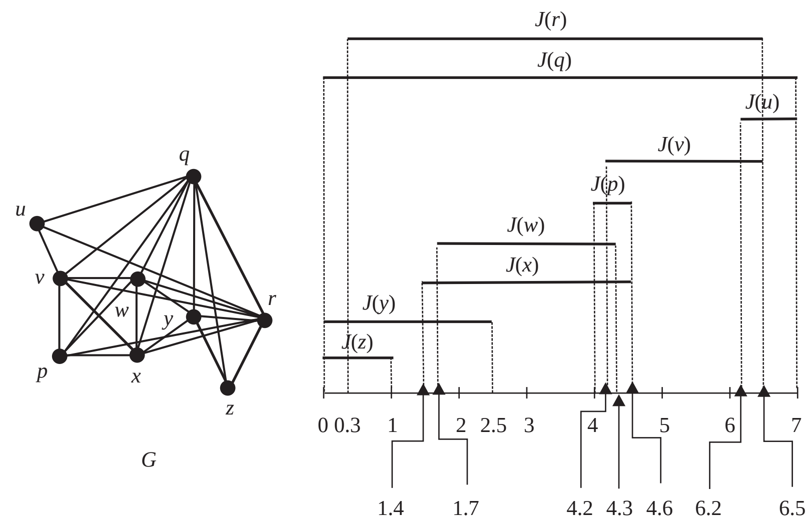

Fig. 4. Interval graph G and an interval assignment to the vertices of G. In this example, $f(u) = (7, 0.8)$, $f(v) = (6.5, 2.8)$, $f(w) = (4.3, 5.3)$, $f(p) = (4.6, 3)$, $f(q) = (7, 7)$, $f(r) = (6.5, 6.7)$, $f(x) = (4.6, 5.6)$, $f(y) = (2.5, 7)$, $f(z) = (1, 7)$, $g(pq) = g(pr) = g(px) = g(pw) = (4, 2.7)$, $g(yz) = g(yq) = g(zq) = (0, 6)$, $g(uq) = g(ur) = g(uv) = (6.2, 0.5)$, $g(vq) = g(vr) = g(vx) = g(vw) = g(pv) = (4.2, 2.7)$, $g(wq)=g(wr)=g(wx)=g(wy)=(1.7, 4.5)$, $g(rq)=g(yr)=g(zr)=(0.3, 6)$, $g(xq)=g(xr)=g(xy)=(1.4, 4.5)$.

We denote by f the above one-to-one function from $V(G)$ to $\mathbb{R}^2$. Now, given an edge vw of G, we define $m(vw)$ by

$$m(vw) = \begin{cases} \min\{r(z) | r(z) \in J(v) \cap J(w), z \in V(G)\} & \text{if there is a vertex } z \text{ in } V(G) \text{ such that } r(z) \in J(v) \cap J(w), \\ M & \text{otherwise.} \end{cases}$$

Then we assign vw an ordered pair

$$(\max\{l(v), l(w)\}, M - \min\{r(v), r(w), m(vw)\}).$$

We denote by g the above function from $E(G)$ to $\mathbb{R}^2$. (See Fig. 4 for an illustration.)

Let $\Im$ be the set $f(V(G)) \cup g(E(G))$. We shall show that the competition graph of the doubly partial order D on $\Im$ is G together with some isolated vertices.

We first show that any vertex in $g(E(G))$ is isolated in $C(D)$. Suppose that there is an arc from $g(vw)$ to $f(x)$. Then $r(x) < \max\{l(v), l(w)\}$ and $\min\{r(v), r(w), m(vw)\} < l(x)$. Since $l(x) < r(x)$, $\min\{r(v), r(w), m(vw)\} < \max\{l(v), l(w)\}$, which is a contradiction. Thus, there is no arc from a vertex in $g(E(G))$ to a vertex in $f(V(G))$.

Now take two vertices $g(vw)$ and $g(xy)$ in $g(E(G))$. Suppose that $\max\{l(v), l(w)\} < \max\{l(x), l(y)\}$. If $\max\{l(x), l(y)\} \in J(v) \cap J(w)$, then

$$\min\{r(v), r(w), m(vw)\} \leqslant \min\{r(x), r(y), m(xy)\}.$$

For, otherwise, let

$$r(z^*) = \min\{r(x), r(y), m(xy)\}.$$

Then

$$r(z^*) < \min\{r(v), r(w), m(vw)\}$$

by the hypothesis. Therefore $r(z^*) < \min\{r(v), r(w)\}$. Since $\max\{l(x), l(y)\} < r(z^*)$ and $\max\{l(v), l(w)\} < \max\{l(x), l(y)\}$, it is true that $\max\{l(v), l(w)\} < r(z^*)$ and so $r(z^*) \in J(v) \cap J(w)$. Then $\min\{r(v), r(w), m(vw)\} \leqslant r(z^*)$ and we reach a contradiction.

If $\max\{l(x), l(y)\} \notin J(v) \cap J(w)$, then $\min\{r(v), r(w)\} \leqslant \max\{l(x), l(y)\}$ and

$$\begin{aligned}\min\{r(v), r(w), m(vw)\} &\leqslant \min\{r(v), r(w)\}\\ &\leqslant \max\{l(x), l(y)\} < \min\{r(x), r(y), m(xy)\}.\end{aligned}$$

In both cases, we have shown that there is no arc from $g(vw)$ to $g(xy)$. Since $g(vw)$ and $g(xy)$ are arbitrarily chosen, we can conclude that there is no arc going from a vertex to another in $g(E(G))$.

Hence we have shown that there is no outgoing arc from a vertex in $g(E(G))$ and so any vertex in $g(E(G))$ is isolated.

We take any two vertices of v and w in $V(G)$. We first suppose that v and w are adjacent in G. Then v and w correspond to ordered pairs

$$f(v) = (r(v), M - l(v))$$

and

$$f(w) = (r(w), M - l(w)),$$

respectively. Since they are adjacent, there exists a vertex

$$g(vw) = (\max\{l(v), l(w)\}, M - \min\{r(v), r(w), m(vw)\})$$

in D. Since v and w are adjacent in G, $J(v)$ and $J(w)$ intersect and therefore $\max\{l(v), l(w)\} < \min\{r(v), r(w)\}$. (Note that $J(v)$ and $J(w)$ are open.) Moreover, $\max\{l(v), l(w)\} < \min\{r(v), r(w), m(vw)\}$. Thus, $g(vw)$ is a common prey of $f(v)$ and $f(w)$ in D.

Now suppose that v and w are non-adjacent in G. Then $J(v) \cap J(w) = \emptyset$ and so either $r(w) \leqslant l(v)$ or $r(v) \leqslant l(w)$. We will show that $f(v)$ and $f(w)$ do not have a common prey. If $f(v)$ does not have a prey, then it is trivially true. Now suppose that $f(v)$ has a prey (a, b). Then $a < r(v)$ and $b < M - l(v)$, and either $(a, b) = g(uy)$ for some $u, y \in V(G)$ or $(a, b) = f(x)$ for some x. Suppose that $(a, b) = g(uy)$ for some $u, y \in V(G)$ and so $a = \max\{l(u), l(y)\}$ and $b = M - \min\{r(u), r(y), m(uy)\}$. Then

$$\max\{l(u), l(y)\} < r(v)$$

and

$$l(v) < \min\{r(u), r(y), m(uy)\}.$$

If $r(w) \leqslant l(v)$, then $r(w) < \min\{r(u), r(y), m(uy)\}$, which implies that $r(w) < \min\{r(u), r(y)\}$. If $a < r(w)$, then $r(w) \in J(u) \cap J(y)$ and $m(uy) \leqslant r(w)$, which is a contradiction. Thus, $a \geqslant r(w)$ and so (a, b) cannot be a prey of $f(w)$. If $r(v) \leqslant l(w)$, then $\max\{l(u), l(y)\} \leqslant l(w)$. Suppose that $l(w) < \min\{r(u), r(y), m(uy)\}$. Then

$$r(v) < \min\{r(u), r(y), m(uy)\}.$$

Since $a < r(v)$, it is true that $r(v) \in J(u) \cap J(y)$ and so

$$\min\{r(u), r(y), m(uy)\} \leqslant r(v),$$

which is a contradiction. Thus, $l(w) \geqslant \min\{r(u), r(y), m(uy)\}$ and so (a, b) cannot be a prey of $f(w)$. Now suppose that $(a, b) = f(x)$ for some $x \in V(G)$. Then $r(x) < r(v)$ and $l(v) < l(x)$. If $r(w) \leqslant l(v)$, then $r(w) < l(x)$ and therefore $r(w) < r(x)$, which implies that (a, b) cannot be a prey of $f(w)$. If $r(v) \leqslant l(w)$, then $r(x) < l(w)$ and therefore $l(x) < l(w)$. Then $M - l(x) > M - l(w)$, which implies that (a, b) cannot be a prey of $f(w)$. Thus, no prey of $f(v)$ can be a prey of $f(w)$. As the situation is symmetric in v and w, any prey of $f(w)$ cannot be a prey of $f(v)$. Hence G together with some isolated vertices added is isomorphic to the competition graph of the doubly partial order D and this completes the proof. □

3. Concluding remarks

Given an interval graph G, Theorem 2 tells us that G is a component of graph G' that contains G together with k isolated vertices and that is the competition graph of a doubly partial order. Given an interval graph G, let $\mathscr{D}(G)$ be the family of acyclic digraphs whose competition graph is G together with k isolated vertices for some k. Finding properties shared by the digraphs in $\mathscr{D}(G)$ will help in characterizing an acyclic digraph D whose competition graph $C(D)$ is interval. We know from Theorem 2 that there is a doubly partial order in $\mathscr{D}(G)$. As the competition number of an interval graph is one, there is an acyclic digraph in $\mathscr{D}(G)$ with $k = 1$. It would be interesting to ask: what is the smallest number k such that the competition graph of a doubly partial order in $\mathscr{D}(G)$ is G together with k added isolated vertices? To answer this question, we define the *doubly partial order competition number* $p(G)$ of an interval graph G by the smallest number p such that G together with p added isolated vertices is isomorphic to the competition graph of a doubly partial order. It is analogous to the notion of the competition number $k(G)$ of a graph G that is the smallest number k such that G together with k added isolated vertices is the competition graph of an acyclic digraph. (Answering Roberts' first question, that of characterizing competition graphs of acyclic digraphs, leads to computing the competition number of a graph.) It is interesting to find the doubly partial order competition number of various interval graphs. It is obvious that the doubly partial order competition number of a complete graph with at least two vertices is the same as its competition number. However, the doubly partial order competition number of path xyz of length two has to be greater than one. Suppose that D is a doubly partial order whose competition number is isomorphic to xyz together with some isolated vertices. Suppose x, y, z correspond to (x_1, x_2), (y_1, y_2), (z_1, z_2), respectively. Then either $x_1 \leqslant z_1$ and $x_2 \geqslant z_2$ or $x_1 \geqslant z_1$ and $z_2 \leqslant z_2$, for, otherwise, a prey of one is that of the

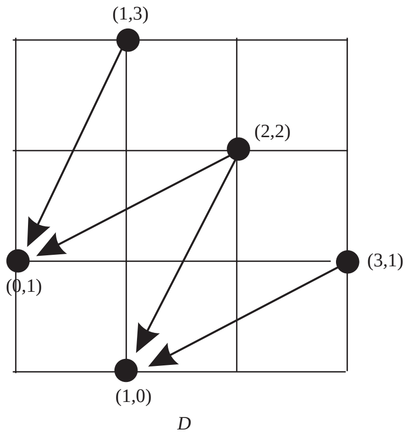

Fig. 5. Doubly partial order D whose competition graph is P_3 together with two isolated vertices.

other and so x and z are adjacent in $C(D)$, a contradiction. This implies that a common prey of (x_1, x_2) and (y_1, y_2) and a common prey of(y_1, y_2) and (z_1, z_2) which are distinct cannot be any of x, y, or z. Thus, the doubly partial order competition number of P_3 of length 2 is two as shown in Fig. 5. It is not difficult to show that the doubly partial order competition number of a path P_n is $n - 1$.

It is also interesting to ask: is there always an acyclic interval digraph in $\mathscr{D}(G)$? (See [20] for the definition of an interval digraph.) Do these digraphs in $\mathscr{D}(G)$ have any properties in common? We have shown that there is a doubly partial order with $k = n - 1$ in $\mathscr{D}(P_n)$. How does this digraph compare to a digraph that we obtain by adding one isolated vertex?

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